CHAPTER I: CARTESIAN COORDINATE SYSTEM

Topics Covered:

The Cartesian XY-plane, Function Graphs, Geometric Shapes, Polygonal Shapes, Areas of Shapes, Theorem of Pythagoras in 2D, Coordinates, Theorem of Pythagoras in 3D, 3D Polygons, Euler's Rule.

Mathematics for Computer Graphics

Computer graphics contains many areas of specialism, such as data visualization, 2D computer animation, film special effects, computer games and 3D computer animation.

Fortunately, not everyone working in computer graphics requires a knowledge of mathematics, but those that do, often look for a book that introduces them to some basic ideas of mathematics, without turning them into mathematicians. This is the objective of this book. Over the following 18 chapters I introduce the reader to some useful mathematical topics that will help them understand the software they work with, and how to solve a wide variety of geometric and algebraic problems.

These topics include numbers systems, algebra, trigonometry, 2D and 3D geometry, vectors, equations, matrices, determinants and calculus.

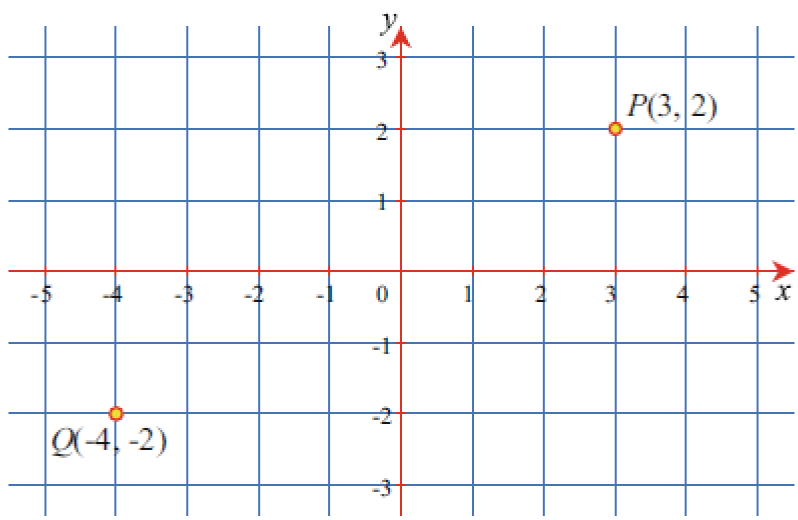
The Cartesian Plane

The Cartesian plane provides a mechanism for locating points with a unique, ordered pair of numbers *(x, y)* as shown in Fig. 5.1,where *P* has coordinates *(*3*,* 2*)* and *Q* has coordinates *(*−4*,*−2*)*. The point *(*0*,* 0*)* is called the *origin*. As previously mentioned,

Descartes suggested that the letters *x* and *y* should be used to represent variables, and letters at the other end of the alphabet should stand for numbers. Which is why equations such as $y = ax^2 + bx + c$,, are written this way.

The axes are said to be *oriented* as the *x*-axis rotates anticlockwise towards the *y*-axis. They could have been oriented in the opposite sense, with the *y*-axis rotating anticlockwise towards the *x*-axis.

Fig. 5.1 The Cartesian plane



Function Graphs

When functions such as

$$
\begin{aligned}
\text{linear:} \quad & y = mx + c, \\
\text{quadratic:} \quad & y = ax^2 + bx + c, \\
\text{cubic:} \quad & y = ax^3 + bx^2 + cx + d, \\
\text{trigonometric:} \quad & y = a \sin x,
\end{aligned}
$$

are drawn as graphs, they create familiar shapes that permit the function to be easily identified. Linear functions are straight lines; quadratics are parabolas; cubics havean 'S' shape; and trigonometric functions often possess a wave-like trace. Figure 5.2

shows examples of each type of function.

Fig. 5.2 Graphs of four function types

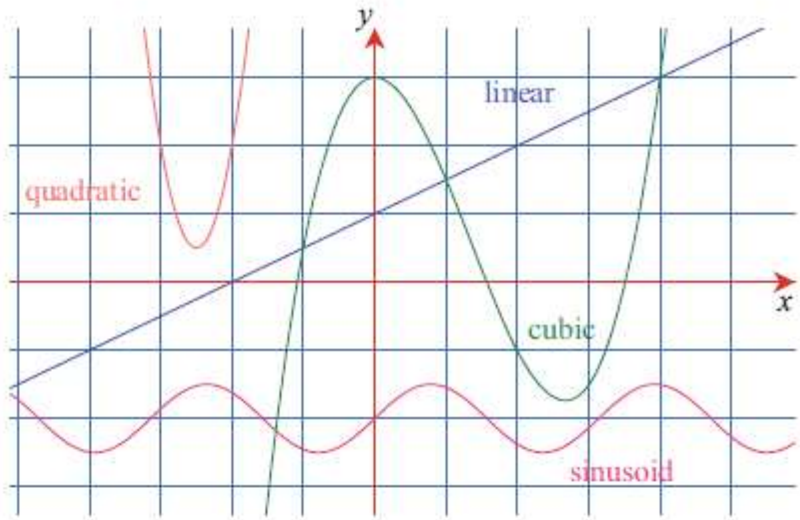
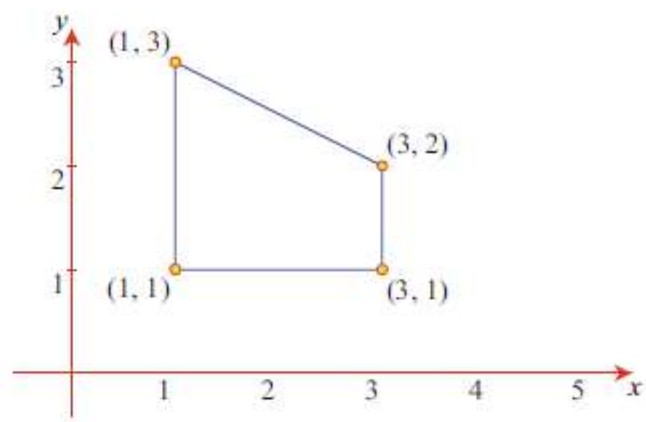


Fig. 5.3 A simple polygon created by a chain of vertices



Shape Representation

The Cartesian plane also provides a way to represent 2D shapes numerically, which permits them to be manipulated mathematically. Let's begin with 2D polygons and show how their internal area can be calculated.

5.5.1 2D Polygons

A polygon is formed from a chain of *vertices* (points) as shown in Fig. 5.3. A straight line is assumed to connect each pair of neighbouring vertices; intermediate points on the line are not explicitly stored. There is no convention for starting a chain of vertices, but software will often dictate whether polygons have a clockwise or anticlockwise vertex sequence.

We can now subject this list of coordinates to a variety of arithmetic and mathematical operations. For example, if we double the values of *x* and *y* and redraw the vertices, we discover that the shape's geometric integrity is preserved, but its size is doubled relative to the origin. Similarly, if we divide the values of *x* and *y* by 2, the shape is still preserved, but its size is halved relative to the origin. On the other

Table 5.1 A polygon's coordinates

x	y
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3

hand, if we add 1 to every *x*-coordinate, and 2 to every *y*-coordinate, and redraw the vertices, the shape's size remains the same but is displaced 1 unit horizontally and 2 units vertically.

Area of a Shape

The area of a polygonal shape is readily calculated from its list of coordinates. For example, using the list of coordinates shown in Table 5.1: the area is computed by

$$area = \frac{1}{2}[(x_0y_1 - x_1y_0) + (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_0 - x_0y_3)].$$

You will observe that the calculation sums the results of multiplying an *x* by the next *y*, minus the next *x* by the previous *y*. When the last vertex is selected, it is paired with the first vertex to complete the process. The result is then halved to reveal the area. As a simple test, let's apply this formula to the shape described in Fig. 5.3:

$$area = \frac{1}{2}[(1 \times 1 - 3 \times 1) + (3 \times 2 - 3 \times 1) + (3 \times 3 - 1 \times 2) + (1 \times 1 - 1 \times 3)]$$

$$area = \frac{1}{2}[-2 + 3 + 7 - 2] = 3.$$

which, by inspection, is the true area. The beauty of this technique is that it works with any number of vertices and any arbitrary shape.

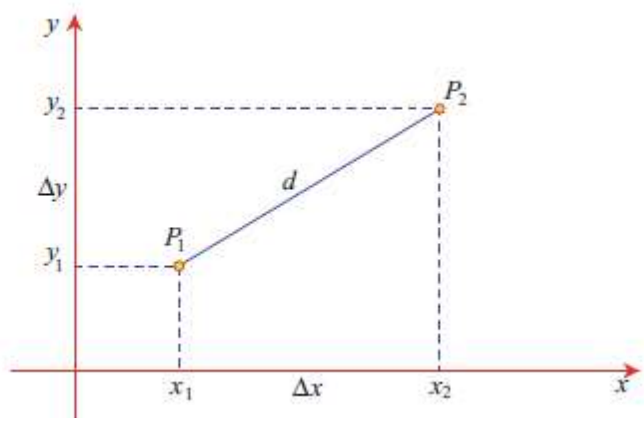
Another feature of the technique is that if the set of coordinates is clockwise, the area is negative, which means that the calculation computes vertex orientation as well as area. To illustrate this feature, the original vertices are reversed to a clockwise sequence as follows:

$$area = \frac{1}{2}[(1 \times 3 - 1 \times 1) + (1 \times 2 - 3 \times 3) + (3 \times 1 - 3 \times 2) + (3 \times 1 - 1 \times 1)]$$

$$area = \frac{1}{2}[2 - 7 - 3 + 2] = -3.$$

The minus sign confirms that the vertices are in a clockwise sequence.

Fig. 5.4 Calculating the distance between two points



Theorem of Pythagoras in 2D

The theorem of Pythagoras is used to calculate the distance between two points.

The theorem of Pythagoras is used to calculate the distance between two points. Figure 5.4 shows two arbitrary points $P_1(x_1, \ y_1)$ and $P_2(x_2, \ y_2)$. The distance $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$. Therefore, the distance d between P_1 and P_2 is given by

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

For example, given $P_1(1, 1), \ P_2(4, 5)$, then $d = \sqrt{3^2 + 4^2} = 5$.

3D Cartesian Coordinates

Two coordinates are required to locate a point on the 2D Cartesian plane, and three coordinates are required for 3D space. The corresponding axial system requires three mutually perpendicular axes; however, there are two ways to add the extra *z*-axis.

Figure 5.5 shows the two orientations, which are described as *left*- and *right-handed* axial systems. The left-handed system permits us to align our left hand with the axes such that the thumb aligns with the *x*-axis, the first finger aligns with the *y*-axis, and the middle finger aligns with the *z*-axis. The right-handed system permits the same system of alignment, but using our right hand. The choice between these axial systems is arbitrary, but one should be aware of the system employed by commercial computer graphics packages. The main problem arises when projecting 3D points onto a 2D plane, which has an oriented axial system. A right-handed system is employed throughout this book, as shown in Fig. 5.6, which also shows a point *P* with its coordinates. It also worth noting that handedness has no meaning in spaces with 4 dimensions or more.

Fig. 5.5 **a** A *left-handed* system. **b** A *right-handed* system

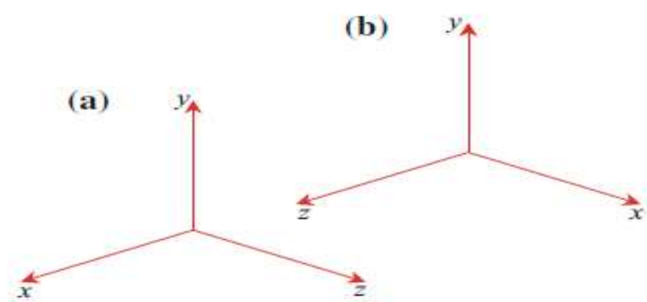
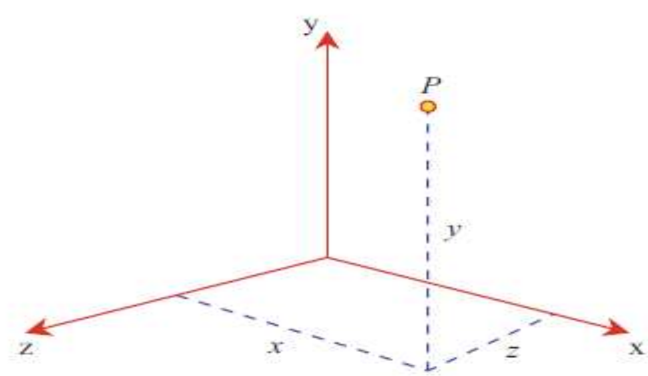


Fig. 5.6 A *right-handed* axial system showing the coordinates of a point *P*



Theorem of Pythagoras in 3D

The theorem of Pythagoras in 3D is a natural extension of the 2D rule. In fact, it even works in higher dimensions.

The theorem of Pythagoras in 3D is a natural extension of the 2D rule. In fact, it even works in higher dimensions. Given two arbitrary points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, we compute $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$ and $\Delta z = z_2 - z_1$, from which the distance d between P_1 and P_2 is given by

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

and the distance from the origin to a point $P(x, y, z)$ is simply

$$d = \sqrt{x^2 + y^2 + z^2}.$$

Therefore, the point (3, 4, 5) is $\sqrt{3^2 + 4^2 + 5^2} \approx 7.07$ from the origin.

Polar Coordinates

Polar coordinates are used for handling data containing angles, rather than linear offsets. Figure 5.7 shows the convention used for 2D polar coordinates, where the

point $P(x, y)$ has equivalent polar coordinates $P(\rho, \theta)$, where:

$$\begin{aligned}
x &= \rho \cos\theta \\
y &= \rho \sin\theta \\
\rho &= \sqrt{x^2 + y^2} \\
\theta &= \arctan(y/x).
\end{aligned}$$

For example, the point $Q(4, 0.8\pi)$ in Fig. 5.7 has Cartesian coordinates:

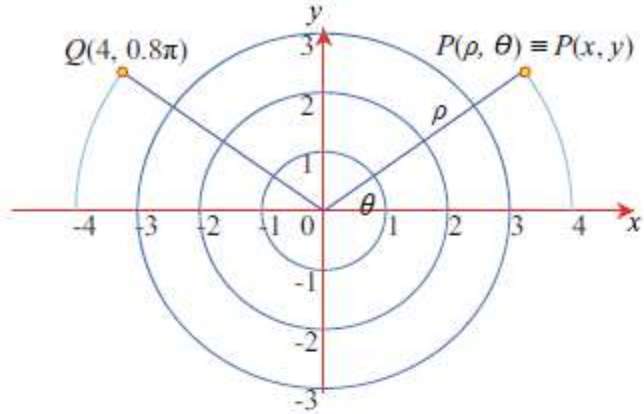
$$\begin{aligned}
x &= 4\cos(0.8\pi) \approx -3.24 \\
y &= 4\sin(0.8\pi) \approx 2.35
\end{aligned}$$

and the point (3, 4) has polar coordinates:

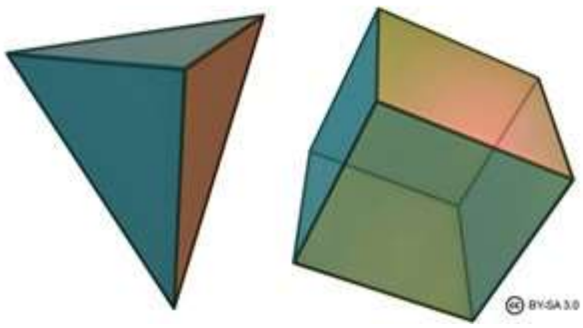
$$\begin{aligned}
\rho &= \sqrt{3^2 + 4^2} = 5 \\
\theta &= \arctan(4/3) \approx 53.13°.
\end{aligned}$$

These conversion formulae work only for the first quadrant. The atan2 function should be used in a software environment, as it works with all four quadrants.

Fig. 5.7 2D polar coordinates

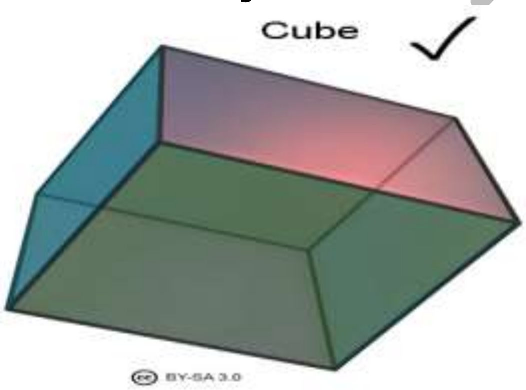


Euler's Formula



What is great about Euler's formula is that it can be understood by almost anyone, as it is so simple to write down. Euler's formula can be understood by someone in Year 7, but is also interesting enough to be studied in universities as part of the mathematical area called topology.

Euler's formula deals with shapes called Polyhedra. A Polyhedron is a closed solid shape which has flat faces and straight edges. An example of a polyhedron would be a cube, whereas a cylinder is not a polyhedron as it has curved edges.

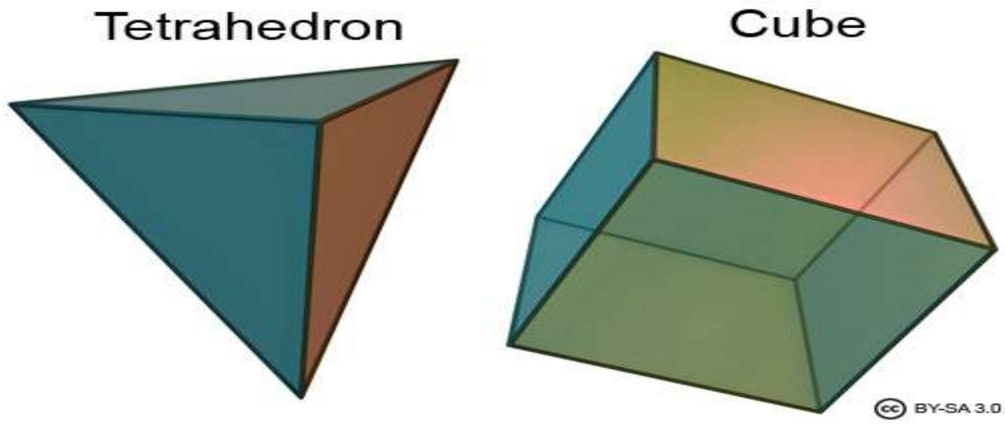


Euler's formula states that for Polyhedra that follow certain rules:

F+V-E=2

Where F= Number of faces, V= Number of vertices (Corners), E=Number of edges

Let us check that this works for the Tetrahedron (triangle based pyramid) and the cube.



Shape	Faces	Vertices	Edges	F+V-E
Tetrahedron	4	4	6	4+4-6=2
Cube	6	8	12	6+8-12=2

Euler's Formula does however only work for Polyhedra that follow certain rules. The rule is that the shape must not have any holes, and that it must not intersect itself. (Imagine taking two opposite faces on a shape and gluing them together at a particular point. This is not allowed.) It also cannot be made up of two pieces stuck together, such as two cubes stuck together by one vertex. If none of these rules are broken, then F+V-E=2 for all Polyhedra. Euler's formula works for most of the common polyhedra which we have heard of.

There are in fact shapes which produce a different answer to the sum F+V-E. The answer to the sum F+V-E is called the Euler Characteristic χ, and is often written $F+V-E=\chi$. Some shapes can even have an Euler Characteristic which is negative! This is where it can all start to get quite complicated. If you want to find about the Euler Characteristic of some of the strange shapes below, why not take a look here.

Who was Euler?

Leonhard Euler (1707-1783) was a Swiss mathematician who is thought to be one of the greatest and most productive mathematicians of all time. Euler spent much of his career blind, but losing his sight only seemed to make him even more productive and at one point he was writing one paper per week, with scribes writing his work down for him. There is more than one formula named after Euler, and the one we have just looked at is sometimes called Euler's Polyhedral formula. Euler's other formula is in the field of complex numbers. Euler is pronounced 'Oiler'.